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Temperature field in random conditions

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Abstract—A probabilistic finite-element approach for modelling the temperature field in structures is proposed. The theoretical formulation of the problem is described. It presents probabilistic distributions for temperature taking into account the random thermal properties of material. An example of the thermal analysis in random conditions is demonstrated.

INTRODUCTION

THE ANALYSIS of thermal problems in structures leads to the building of mathematical models and representing external disturbances. For those designing important structures the consequences of thermal damage makes a probabilistic approach inescapable. The application of probabilistic processes to the representation of thermal boundary conditions and the use of finite elements to model structures could be a convenient tool for thermal analysis. The literature on the probabilistic methods in mechanics for problems involving time-independent uncertainties is quite considerable and so only a few sample references are indicated here [1–4]. It can be classified into two major categories, i.e. methods using a statistical approach and those using a non-statistical approach. Non-statistical approaches include stochastic finite element methods [5–7]. The probabilistic methods for the non-statistical analysis of structural and continuum problems have been discussed in refs. [8, 9]. The probabilistic methods for the transient heat flow analysis of random field problems by the finite element method have not been undertaken. This paper describes the theoretical aspects of the temperature field with random material properties and boundary conditions in the case when the problem of heat flow is formulated in terms of finite elements. Heat transfer phenomena described by matrix equations based on the Galerkin method are widely used [10–12] and their formulations can be found in general textbooks on finite element analysis [13].

HEAT FLOW EQUATIONS

The variation of temperature T with time t in a two-dimensional region Ω, relative to the Cartesian coordinates $\mathbf{x}$ is governed by the equation

$$\rho c \frac{\partial T}{\partial t} = \nabla(\lambda \nabla T) + q \tag{1}$$

where λ is the conductivity tensor, c the heat capacity, ρ the density, and q the rate of heat generation. At the surface of the body the temperature may be prescribed or the flow of heat due to convection or radiation may exist. The region Ω is divided into a number of finite elements Ω^e with shape function N^i associated with each node i. The unknown function T is approximated through the solution domain at any time t by

$$T = \sum_{i=1}^{N} N^i T^i(t) = \mathbf{NT} \tag{2}$$

where $\mathbf{T}$ is the column vector of nodal values T^i.

The substitution of expansion (2) into equation (1) and the application of the Galerkin method produce the following equation:

$$\underset{\sim}{C}\dot{\mathbf{T}} + \underset{\sim}{K}\mathbf{T} = \mathbf{F}. \tag{3}$$

The form of the matrices $\underset{\sim}{C}$, $\underset{\sim}{K}$ and the vector $\mathbf{F}$ together with a description of the temporal discretization of equation (3) have been described by many authors (see refs. [10–12] for instance) and will not be considered further.

FORMULATION OF THE PROBLEM

We consider equation (3) in which the matrices $\underset{\sim}{C}$, $\underset{\sim}{K}$ and the vectors $\mathbf{F}$ and $\mathbf{T}$ are functions of the discretized random variable vector $\mathbf{b} = \mathbf{b}(\mathbf{x})$

$$\underset{\sim}{C}(\mathbf{b})\dot{\mathbf{T}}(\mathbf{b}, t) + \underset{\sim}{K}(\mathbf{b})\mathbf{T}(\mathbf{b}, t) = \mathbf{F}(\mathbf{b}, t). \tag{4}$$

The random function $b(\mathbf{x})$ is approximated using shape functions $N_i(x)$ by

$$b(\mathbf{x}) = \sum_{i=1}^{q} N_i(x) b_i = \mathbf{Nb} \tag{5}$$

where b_i are the nodal values of $b(\mathbf{x})$, that is the values of b at x_i, $i = 1, \ldots, q$.

The mean value of $\mathbf{b}$ denoted by $E(\mathbf{b})$ is expressed as

$$E(\mathbf{b}) = \sum_{i=1}^{q} N_i E(b_i) \tag{6}$$

and the variance by

$$V(\mathbf{b}) = \alpha^2 E(\mathbf{b})^2 \tag{7}$$

where α is the coefficient of variation.

All the random functions are expanded about the mean value $E(\mathbf{b})$ via a Taylor series and only up to second-order terms are retained. For any small parameter γ we have

$$\mathbf{T}(\mathbf{b},t) = E(\mathbf{T}(t)) + \gamma \sum_{i=1}^{q} E(\mathbf{T}_{b_i}(t))\Delta b_i + \tfrac{1}{2}\gamma^2 \sum_{i,j=1}^{q} E(\mathbf{T}_{b_i b_j}(t))\Delta b_i \Delta b_j \tag{8}$$

where Δb_i represents the first-order variation of b_i about $E(b_i)$ and for any function

$$E(g(\mathbf{x})) = g(x, E(b)) \tag{9}$$

$$E(g_{b_1}) = \frac{\partial g}{\partial b_1} \tag{10}$$

$$E(g_{b_1 b_2}) = \frac{\partial^2 g}{\partial b_1 \partial b_2}. \tag{11}$$

In a similar manner we can express $\underset{\approx}{C}(\mathbf{b})$, $\underset{\approx}{K}(\mathbf{b})$ and $\mathbf{F}(\mathbf{b},t)$ as

$$\underset{\approx}{C}(\mathbf{b}) = E(\underset{\approx}{C}) + \gamma \sum_{i=1}^{q} E(\underset{\approx}{C}_{b_i})\Delta b_i + \tfrac{1}{2}\gamma^2 \sum_{i,j=1}^{q} E(\underset{\approx}{C}_{b_i b_j})\Delta b_i \Delta b_j \tag{12}$$

$$\underset{\approx}{K}(\mathbf{b}) = E(\underset{\approx}{K}) + \gamma \sum_{i=1}^{q} E(\underset{\approx}{K}_{b_i})\Delta b_i + \tfrac{1}{2}\gamma^2 \sum_{i,j=1}^{q} E(\underset{\approx}{K}_{b_i b_j})\Delta b_i \Delta b_j \tag{13}$$

$$\mathbf{F}(\mathbf{b},t) = E(\mathbf{F}) + \gamma \sum_{i=1}^{q} E(\mathbf{F}_{b_i}(t))\Delta b_i + \tfrac{1}{2}\gamma^2 \sum_{i,j=1}^{q} E(\mathbf{F}_{b_i b_j})\Delta b_i \Delta b_j. \tag{14}$$

Substitution of equations (8) and (12)–(14) into equation (3) and collecting terms of order 1, γ and γ^2 yields the following equations for $E(\mathbf{T}(t))$, $E(\mathbf{T}_{b_i}(t))$ and $E(\mathbf{T}_{b_i b_j}(t))$:

zeroth order

$$E(\underset{\approx}{C})E(\dot{\mathbf{T}}(t)) + E(\underset{\approx}{K})E(\mathbf{T}(t)) = E(\mathbf{F}(t)); \tag{15}$$

first-order

$$E(\underset{\approx}{C})E(\dot{\mathbf{T}}_{b_i}(t)) + E(\underset{\approx}{K})E(\mathbf{T}_{b_i}(t)) = E(\mathbf{F}_{1b_i}(E(\mathbf{T}),t)) \tag{16}$$

where

$$E(\mathbf{F}_{1b_i}(E(\mathbf{T}),t)) = E(\mathbf{F}_{b_i}(t)) - (E(\underset{\approx}{C}_{b_i})E(\dot{\mathbf{T}}(t)) + E(\underset{\approx}{K}_{b_i})E(\mathbf{T}(t))); \tag{17}$$

second order

$$E(\underset{\approx}{C})\hat{\dot{\mathbf{T}}}_2(t) + E(\underset{\approx}{K})\hat{\mathbf{T}}_2(t) = F_2(E(\mathbf{T}), E(\mathbf{T}_{b_i}), t) \tag{18}$$

$$\begin{aligned}\hat{\mathbf{F}}_2 = &\sum_{i,j=1}^{q} \{[\tfrac{1}{2}E(\mathbf{F}_{b_i b_j}(t))]\operatorname{cov}(b_i,b_j)\} \\ &- \sum_{i,j=1}^{q} \{[\tfrac{1}{2}E(\underset{\approx}{C}_{b_i b_j})E(\dot{\mathbf{T}}(t)) + \tfrac{1}{2}E(\underset{\approx}{K}_{b_i b_j})E(\mathbf{T}(t)) \\ &+ E(\underset{\approx}{C}_{b_i})E(\dot{\mathbf{T}}_{b_j}(t)) + E(\underset{\approx}{K}_{b_i})E(\mathbf{T}_{b_j}(t))]\operatorname{cov}(b_i,b_j)\}\end{aligned} \tag{19}$$

and

$$\hat{\mathbf{T}}_2(t) = \frac{1}{2}\sum_{i,j=1}^{q} E(\mathbf{T}_{b_i b_j})\operatorname{cov}(b_i,b_j) \tag{20}$$

$$\operatorname{Cov}(b_i,b_j) = [V(b(x_i))V(b(x_j))]^{1/2}R(b(x_i),b(x_j)) \tag{21}$$

and $R(b(x_i),b(x_j))$ is the autocorrelation.

EXPECTATION VALUES OF TEMPERATURE

The definitions for the expectation and autocovariance of the temperature are given by

$$E(\mathbf{T}) = \int_{-\infty}^{\infty} \mathbf{T}(\mathbf{b},t)f(b)\,\mathrm{d}\mathbf{b} \tag{22}$$

and

$$\operatorname{Cov}(T^i,T^j) = \int_{-\infty}^{\infty} (T^i - E(T^i))(T^j - E(T^j))f(\mathbf{b})\,\mathrm{d}\mathbf{b} \tag{23}$$

where $f(\mathbf{b})$ is the joint probability density function. The second-order estimate of the mean value of $\mathbf{T}$ is obtained from equation (22) to give

$$E(\mathbf{T}) = \mathbf{T}(E(\mathbf{b})) + \tfrac{1}{2}\left\{\sum_{i,j=1}^{q} E(\mathbf{T}_{b_i b_j})\operatorname{Cov}(b_i,b_j)\right\}. \tag{24}$$

The thermal conductivity matrix $\underset{\approx}{K}^e$ for an element e can be expressed as [10–12]

$$E(\underset{\approx}{K}^e) = \int_{\Omega^e} \underset{\approx}{B}^e(\mathbf{x})\lambda^e(\mathbf{x},\mathbf{b})\underset{\approx}{B}^e(\mathbf{x})\,\mathrm{d}\Omega^e \tag{25}$$

where the matrix $\underset{\approx}{B}^e$ describes the temperature gradients within the element as a function of nodal temperatures

$$\nabla T = \underset{\approx}{B}(\mathbf{x})\mathbf{T} \tag{26}$$

and

$$E(\underset{\approx}{C}^e) = \int_{\Omega^e} C(\mathbf{x},\mathbf{b})\mathbf{N}^e\mathbf{N}^e\,\mathrm{d}\Omega^e \tag{27}$$

where $C = c \cdot \rho$

$$E(\mathbf{F}^e) = \int_{\Omega^e} \mathbf{N}^e\mathbf{q}\,\mathrm{d}\Omega. \tag{28}$$

REDUCTION OF THE FORM OF THE GOVERNING EQUATIONS

Consider the correlated full covariance matrix

$$\mathrm{Cov}\,(b_i, b_j) = B_{ij} \qquad (29)$$

and an uncorrelated diagonal variance matrix

$$V(c_i, c_j) = V(c_i)\delta_{ij} = \Gamma_{ij} \qquad (30)$$

where $\underset{\approx}{\Gamma}$ is a diagonal matrix and δ_{ij} the Kronecker delta. Transformation of matrix $\underset{\approx}{B}$ to matrix $\underset{\approx}{\Gamma}$ can be expressed by the eigenproblem

$$\underset{\approx}{B}\underset{\approx}{N} = \underset{\approx}{N}\underset{\approx}{\Gamma} \qquad (31)$$

where the matrix $\underset{\approx}{N}$ possesses the following property:

$$\underset{\approx}{N}^{\mathrm{T}}\underset{\approx}{N} = \underset{\approx}{N}\underset{\approx}{N}^{\mathrm{T}} = \underset{\approx}{I}_q \qquad (32)$$

where $\underset{\approx}{I}_q$ is the identity matrix and superscript T denotes the transpose.

The transformed random variable **c** can be given by

$$\mathbf{c} = \underset{\approx}{N}^{\mathrm{T}}\mathbf{b} \qquad (33)$$

or

$$c_i = \sum_{j=1}^{q} N_{ji} b_j. \qquad (34)$$

The mean and variance of **c** are given by

$$E(\mathbf{c}) = \underset{\approx}{N}^{\mathrm{T}}E(\mathbf{b}) \qquad (35)$$

and

$$V(c_i) = \Gamma_{ii}.$$

An introduction of an uncorrelated random vector **c** reduces the zeroth-, first-, and second-order equations given by equations (15)–(20) to the following form:

zeroth order

$$E(\underset{\approx}{C})E(\dot{\mathbf{T}}(t)) + E(\underset{\approx}{K})E(\mathbf{T}(t)) + E(\mathbf{F}(t)) = 0; \qquad (36)$$

first order

$$E(\underset{\approx}{C})E(\dot{\mathbf{T}}_{c_i}(t)) + E(\underset{\approx}{K})E(\mathbf{T}_{c_i}(t)) + E(\mathbf{F}_{1c_i}(E(\mathbf{T}), t)) = 0, \quad i = 1, \ldots, r$$

where

$$E(\mathbf{F}_{1c_i}(E(\mathbf{T}), t)) = E(\mathbf{F}_{c_i}(t)) - [E(\underset{\approx}{C}_{c_i})E(\dot{\mathbf{T}}(t)) + E(\underset{\approx}{K}_{c_i})E(\mathbf{T}(t))], \quad i = 1, \ldots, r;$$

second order

$$E(\underset{\approx}{C})E(\dot{\mathbf{T}}_2(t)) + E(\underset{\approx}{K})E(\mathbf{T}_2(t)) + E(\mathbf{F}_2(E(\mathbf{T}), E(\mathbf{T}_{c_i}), t)) = 0$$

where

$$E(\mathbf{F}_2(E(\mathbf{T}), E(\mathbf{T}_{c_i}), t)) = \sum_{i=1}^{r} \{(\tfrac{1}{2}E(\mathbf{F}_{c_i c_i}(t)))V(c_i)\} - \sum_{i=1}^{r} \{[\tfrac{1}{2}E(\underset{\approx}{C}_{c_i c_i})E(\dot{\mathbf{T}}(t)) + \tfrac{1}{2}E(\underset{\approx}{K}_{c_i c_i})E(\dot{\mathbf{T}}_{c_i}(t)) + E(\underset{\approx}{K}_{c_i})E(\mathbf{T}_{c_i}(t))]V(c_i)\}$$

and

$$E(\mathbf{T}_2(t)) = \tfrac{1}{2}\sum_{i=1}^{r} E(\mathbf{T}_{c_i c_i}(t))V(c_i).$$

EXAMPLE

The method developed in the above sections has been tested by studying the temperature field in the rectangular region undergoing external heat rates. A two-dimensional region of the analysis is presented in Fig. 1. Assume the following boundary conditions: 11 random variables on the upper boundary surface (see Fig. 1) and $\partial T/\partial n = 0$ on the other boundary sides of the region. The domain is divided by four-nodal isoparametric elements.

Geometrical and material properties are as follows: $L = 10$ cm, heat capacity $c = 1$ J g^{-1} K^{-1}, thermal conductivity $k = 1$ W cm^{-1} K^{-1}, density $\rho = 1$ g cm^{-3}.

Random heat sources are assumed as follows: 11 random variables, coefficient of variation 0.1, mean heat rate $\dot{q}_1 = 1$ W cm^{-3}, spatial correlation $R(x_i, x_j) = \exp(-\mathrm{abs}(x_i - x_j)/6L)$.

Random material: 3 random variables, coefficient of variation 0.1, spatial correlation $R(x_i, x_j) = \exp(-\mathrm{abs}(x_i - x_j)/1.5L)$. The results of the analysis for point A chosen in the centre of the region are presented in Fig. 2. It is seen that the temperature at this point varies in the interval of temperatures from upper to lower values because of the random variables assumed in the example. In practice temperatures in the analysed regions are functions of random material properties and boundary conditions. The result indicates that changes of material characteristics should be taken into account in the face of design of constructions.

FINAL REMARKS

The development of numerical methods for thermal analysis in continua with random properties requires the unification of mechanics, probability and numerical methods. It is an attractive tool for computation

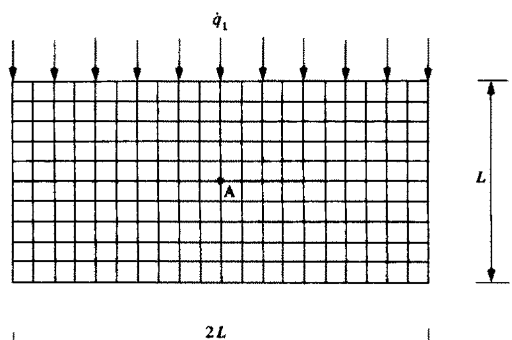

FIG. 1. Region of analysis.

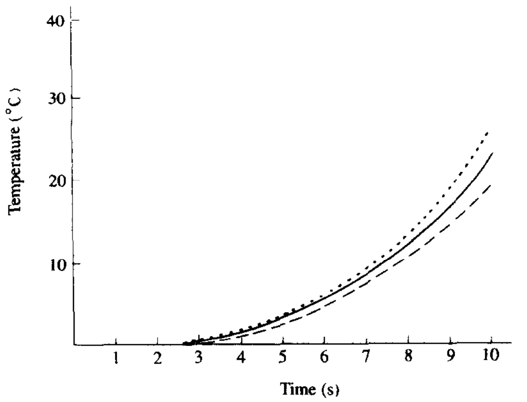

FIG. 2. Temperature distribution at point A : ——, mean value; ······, upper value; – -- –, lower value.

of thermal variables considering random changes in material and boundary conditions. An application of the finite element method to discretization of the region with the heat flow equation is a convenient approach for the model presented.

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CHAMP DE TEMPERATURE DANS DES CONDITIONS ALEATOIRES

Résumé—On propose une approche probabiliste aux éléments finis pour modéliser le champ de température dans des structures. La formulation théorique du problème est décrite. Elle présente les distributions probabilistes de la température en prenant en compte les propriétés thermiques aléatoires des matériaux. On donne un exemple de l'analyse thermique dans des conditions aléatoires

TEMPERATURFELD BEI BELIEBIGEN BEDINGUNGEN

Zusammenfassung—Das Temperaturfeld in Strukturen wird mit Hilfe des Verfahrens der finiten Elemente näherungsweise dargestellt. Die theoretische Beschreibung des Problems wird vorgestellt. Die näherungsweise berechneten Temperaturverteilungen berücksichtigen beliebige thermische Stoffeigenschaften. Es wird ein Beispiel der thermischen Untersuchungen bei beliebigen Bedingungen gezeigt.

ТЕМПЕРАТУРНОЕ ПОЛЕ ПРИ ПРОИЗВОЛЬНО ЗАДАННЫХ УСЛОВИЯХ

Аннотация—Предложен вероятностный метод конечных элементов для моделирования температурного поля в структурах. Описывается теоретическая постановка задачи, включающая распределения температур с учетом случайных тепловых свойств материала. Приводится пример термического анализа при произвольно заданных условиях.